

# Conformal Invariance of Unitarity Corrections \*

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## Abstract

We study perturbative unitarity corrections in the generalized leading logarithmic approximation in high energy QCD. It is shown that the corresponding amplitudes with up to six gluons in the  $t$ -channel are conformally invariant in impact parameter space. In particular we give a new representation for the two-to-six reggeized gluon vertex in terms of conformally invariant functions. With the help of this representation an interesting regularity in the structure of the two-to-four and the two-to-six transition vertices is found.

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# 1 Introduction

One of the most interesting properties of the leading logarithmic approximation (LLA) in the Regge limit of QCD is the conformal invariance of scattering amplitudes in two-dimensional impact parameter space. In LLA the scattering of small color dipoles is described by the BFKL Pomeron [1, 2]. It resums large logarithms of the energy  $\sqrt{s}$  which can compensate the smallness of the strong coupling constant  $\alpha_s$ . After a Fourier transformation from transverse momentum space to two-dimensional impact parameter space the corresponding amplitude is invariant under global conformal transformations, i. e. under Möbius transformations [3]. The perturbative (BFKL) Pomeron describes the exchange of two interacting reggeized gluons in the  $t$ -channel. The unitarization of the scattering amplitude requires to take into account also contributions with larger numbers of gluons in the  $t$ -channel which are consequently called unitarity corrections. In [4] it was shown that also a system of  $n$  interacting reggeized gluons (described by the BKP equations [5, 6]) is invariant under conformal transformations in impact parameter space.

More recently also transitions between states containing different numbers of reggeized gluons were studied [7]–[15]. These transition vertices have been obtained in the so-called generalized leading logarithmic approximation (GLLA) [5, 16, 17]. Amplitudes with up to six reggeized gluons in the  $t$ -channel have been investigated in the GLLA. It was found that these amplitudes have the structure of an effective field theory in which only states with even numbers of gluons occur. These states are coupled to each other via number changing vertices. In particular the two-to-four gluon vertex  $V_{2\rightarrow 4}$  [8, 9] and the two-to-six gluon vertex  $V_{2\rightarrow 6}$  [10] have been calculated explicitly. These two are the only number changing vertices present in the amplitudes with up to six gluons.

The natural question arises whether also the transition vertices exhibit conformal symmetry in impact parameter space. For the case of the two-to-four gluon vertex a positive answer to this question was given in [18]. The proof of conformal invariance of this vertex was later simplified in [13, 14]. In the present paper we will show that also the two-to-six gluon vertex is conformally invariant. We can then further conclude that all amplitudes describing perturbative unitarity corrections with up to six gluons in the  $t$ -channel are conformally invariant. This becomes clear when we recall the field theory structure observed in those amplitudes. The amplitudes in fact contain only elements which are conformally invariant. The amplitude describing the production of three gluons, for example, turns out to be a superposition of two-gluon (BFKL) amplitudes [9]. Similarly, the five-gluon amplitudes is a superposition of two- and four-gluon amplitudes [10] etc. The conformal invariance of the building blocks, i. e. of the  $n$ -gluon states and of the transition vertices, is thus sufficient to prove the conformal invariance of the full amplitudes.

In section 2 we briefly review the BFKL kernel and the two-to-four reggeized

gluon vertex as well as an auxiliary function which is already known to be conformally invariant. In section 3 we then show that the two-to-six reggeized gluon vertex can be represented in terms of this function. The comparison of the new representation for  $V_{2\rightarrow 6}$  with a similar one for  $V_{2\rightarrow 4}$  allows us to observe an interesting regularity in the structure of the number changing vertices.

## 2 The BFKL kernel and the two-to-four gluon vertex

The interaction of two reggeized gluons is described by the BFKL kernel. In impact parameter space it can be represented most conveniently using pseudodifferential operators. One defines complex notation for the two-dimensional gluon coordinates,

$$\vec{\rho} = (\rho_x, \rho_y) \longrightarrow \rho = \rho_x + i\rho_y \quad (1)$$

and corresponding derivatives  $\partial = \frac{\partial}{\partial \rho}$ . The interaction kernel for two gluons with complex coordinates  $\rho_1$  and  $\rho_2$  then reads [19, 20]

$$\mathcal{K} = \frac{g^2 N_c}{8\pi^2} (K + K^*) \quad (2)$$

with

$$K = \log[(\rho_1 - \rho_2)^2 \partial_1] + \log[(\rho_1 - \rho_2)^2 \partial_2] - 2\log(\rho_1 - \rho_2) - 2\psi(1). \quad (3)$$

Here  $\psi$  denotes the logarithmic derivative of the Euler gamma function.

According to (2) the BFKL kernel is a sum of two operators, the first of which acts only on the holomorphic coordinates  $\rho_1, \rho_2$ , whereas the second acts on the antiholomorphic coordinates  $\rho_1^*, \rho_2^*$  only. This property of the BFKL kernel is called holomorphic separability.

With the help of the representation (2) it is relatively easy to show that the kernel  $\mathcal{K}$  is invariant under conformal (Möbius) transformations of the gluon coordinates

$$\rho \longrightarrow \frac{a\rho + b}{c\rho + d}, \quad \rho^* \longrightarrow \frac{a^*\rho^* + b^*}{c^*\rho^* + d^*} \quad (4)$$

with  $ad - bc = a^*d^* - b^*c^* = 1$ . These transformations are thus characterized by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{C})/Z_2, \quad (5)$$

i. e. the group of projective conformal transformations.

Before we consider the two-to-four gluon transition vertex  $V_{2\rightarrow 4}$  we define some useful functions. Let  $\phi_2(\mathbf{k}_1, \mathbf{k}_2)$  denote a two-gluon (BFKL) amplitude,  $\mathbf{k}_i$

being the transverse momenta of the gluons. Then we define

$$a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \int \frac{d^2\mathbf{l}}{(2\pi)^3} \frac{\mathbf{k}_1^2}{(1 - \mathbf{k}_2)^2 [1 - (\mathbf{k}_1 + \mathbf{k}_2)]^2} \phi_2 \left( 1, \sum_{j=1}^3 \mathbf{k}_j - 1 \right), \quad (6)$$

$$b(\mathbf{k}_1, \mathbf{k}_2) = a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 = 0), \quad (7)$$

$$c(\mathbf{k}_1) = b(\mathbf{k}_1, \mathbf{k}_2 = 0), \quad (8)$$

$$s(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = -\frac{2}{N_c g^2} \beta(\mathbf{k}_1) \phi_2(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3), \quad (9)$$

$$t(\mathbf{k}_1, \mathbf{k}_2) = s(\mathbf{k}_1, \mathbf{k}_3 = 0, \mathbf{k}_2), \quad (10)$$

where  $\alpha = 1 + \beta$  is the well-known gluon trajectory function with

$$\beta(\mathbf{k}^2) = -\frac{N_c}{2} g^2 \int \frac{d^2\mathbf{l}}{(2\pi)^3} \frac{\mathbf{k}^2}{\mathbf{l}^2 (1 - \mathbf{k})^2}. \quad (11)$$

These functions are not infrared finite by themselves but will occur only in infrared finite combinations. An important example for such an infrared finite combination is the function  $G$ ,

$$G(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{g^2}{2} [2c(123) - 2b(12, 3) - 2b(23, 1) + 2a(2, 1, 3) + t(12, 3) + t(23, 1) - s(2, 1, 3) - s(2, 3, 1)], \quad (12)$$

$g$  being the gauge coupling. Here we have introduced a shorthand notation for the momentum arguments by replacing the momentum  $\mathbf{k}_i$  by its index  $i$ . A string of indices stands for a sum of the corresponding momenta, for example we have  $t(12, 3) = t(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_3)$ . The function  $G$  describes a transition kernel from the two gluons in the amplitude  $\phi_2$  to three gluons with momenta  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ . The functions  $a, b, c$  correspond to real gluon emission, whereas  $s$  and  $t$  describe virtual corrections. Using the definitions of these functions it is also possible to write  $G = \tilde{G} \phi_2$  with an integral operator  $\tilde{G}$  acting on the two-gluon amplitude  $\phi_2$ . We note that the function  $G(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$  is not symmetric in its three arguments. It vanishes when the first or the last argument vanishes,

$$G(\mathbf{k}_1 = 0, \mathbf{k}_2, \mathbf{k}_3) = G(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 = 0) = 0, \quad (13)$$

but it does not vanish when its second argument  $\mathbf{k}_2$  vanishes. Interestingly, in this case  $G$  happens to reduce (up to a color factor) to a BFKL kernel  $\mathcal{K}$ ,

$$G(\mathbf{k}_1, \mathbf{k}_2 = 0, \mathbf{k}_3) = \frac{1}{N_c} \mathcal{K}(\mathbf{k}_1, \mathbf{k}_3) \otimes \phi_2 \quad (14)$$

$$= \frac{g^2}{2} [2c(13) - 2b(1, 3) - 2b(3, 1) + t(1, 3) + t(3, 1)]. \quad (15)$$

In the unitarity corrections  $G$  never occurs as an isolated object. It should therefore not be confused with one of the transition vertices of the effective field theory

of unitarity corrections. The transition vertices, however, can be expressed in terms of  $\tilde{G}$ , as we will show explicitly for  $V_{2 \rightarrow 4}$  and  $V_{2 \rightarrow 6}$ .

Let us now turn to the two-to-four gluon transition vertex  $V_{2 \rightarrow 4}$ . It couples a two-gluon amplitude  $\phi_2$  to a four-gluon amplitude  $\phi_4$ . We will assume that the two gluons in  $\phi_2$  are in a color singlet state. (The vertex  $V_{2 \rightarrow 4}$  is in fact known only for this situation.) The arguments of  $V_{2 \rightarrow 4}$  are the transverse momenta  $\mathbf{q}_j$  of the two incoming gluons and the transverse momenta  $\mathbf{k}_i$  of the four outgoing gluons. In addition, the transition vertex carries four gluon color labels for the outgoing gluons. The vertex was computed in [8, 9] where it was shown to have the following structure,

$$\begin{aligned} V_{2 \rightarrow 4}^{a_1 a_2 a_3 a_4}(\{\mathbf{q}_j\}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = & \delta_{a_1 a_2} \delta_{a_3 a_4} V(\{\mathbf{q}_j\}; \mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) \\ & + \delta_{a_1 a_3} \delta_{a_2 a_4} V(\{\mathbf{q}_j\}; \mathbf{k}_1, \mathbf{k}_3; \mathbf{k}_2, \mathbf{k}_4) \\ & + \delta_{a_1 a_4} \delta_{a_2 a_3} V(\{\mathbf{q}_j\}; \mathbf{k}_1, \mathbf{k}_4; \mathbf{k}_2, \mathbf{k}_3). \end{aligned} \quad (16)$$

As can be derived from this color and momentum structure, the vertex is completely symmetric in the four outgoing gluons, i.e. under the simultaneous exchange of color labels and momenta of the gluons. The vertex  $V_{2 \rightarrow 4}$  should actually be understood as an integral operator acting on  $\phi_2$ . We will be mainly interested in the combination  $V_{2 \rightarrow 4} \phi_2$ , which is its action on the two-gluon amplitude. The function  $V$  and its action on  $\phi_2$  were first computed in terms of integrals of the form  $a, b, c, s$ , and  $t$ . We will not need this representation explicitly here and give an alternative expression momentarily. We note that the function  $V(\{\mathbf{q}_j\}; \mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4)$  is symmetric in the two momenta  $\mathbf{q}_j$ , in the two momenta  $\mathbf{k}_1$  and  $\mathbf{k}_2$ , as well as in  $\mathbf{k}_3$  and  $\mathbf{k}_4$ , hence their separation in the notation. It is further symmetric under the exchange of the pairs  $\{\mathbf{k}_1, \mathbf{k}_2\}$  and  $\{\mathbf{k}_3, \mathbf{k}_4\}$ .

To explain the conformal invariance of  $V_{2 \rightarrow 4}$  we consider an amplitude  $\mathcal{A}_4$  which is the convolution of the vertex  $V_{2 \rightarrow 4}$  with the BFKL amplitude  $\phi_2$  and a four-gluon state  $\phi_4$ . The latter is assumed to be a solution of the four-particle BKP equation. The amplitude  $\mathcal{A}_4$  has the form

$$\begin{aligned} \mathcal{A}_4 = & \int \prod_{j=1}^2 d^2 \mathbf{q}_j \prod_{i=1}^4 d^2 \mathbf{k}_i \phi_2(\mathbf{q}_1, \mathbf{q}_2) V_{2 \rightarrow 4}^{a_1 a_2 a_3 a_4}(\{\mathbf{q}_j\}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \times \\ & \times \phi_4^{a_1 a_2 a_3 a_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \delta \left( \sum_{j=1}^2 \mathbf{q}_j - \sum_{i=1}^4 \mathbf{k}_i \right). \end{aligned} \quad (17)$$

Transverse momentum space and impact parameter space are related to each other by a Fourier transformation,

$$\phi_2(\mathbf{q}_1, \mathbf{q}_2) = \int \prod_{j=1}^2 [d^2 \rho_{j'} e^{i \mathbf{q}_j \rho_{j'}}] \phi_2(\rho_{1'}, \rho_{2'}) \quad (18)$$

$$\phi_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \int \prod_{i=1}^4 [d^2 \rho_i e^{-i \mathbf{k}_i \rho_i}] \phi_4(\rho_1, \rho_2, \rho_3, \rho_4), \quad (19)$$

where the gluon coordinates  $\rho$  are understood in complex notation as introduced in eq. (1). Applying this procedure to (17) defines the Fourier transform of the transition vertex. Using the conformal invariance of the functions  $\phi_2$  and  $\phi_4$  (see [4]) one can prove the invariance of the whole amplitude  $\mathcal{A}_4$  under a simultaneous Möbius transformation of all gluon coordinates  $\rho_i$  and  $\rho_{j'}$  according to (4), and then infer that the vertex  $V_{2\rightarrow 4}$  is in fact conformally invariant. In [18] this proof was performed using the original representation for  $V\phi_2$  mentioned above. In [13, 14] it was shown that the proof of conformal invariance can be simplified by making use of a representation that can be found already in [9] for the forward direction ( $\sum_i \mathbf{k}_i = 0$ ),

$$\begin{aligned} (V\phi_2)(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4) = & \frac{g^2}{2} [G(12, -, 34) \\ & - G(12, 3, 4) - G(12, 4, 3) - G(1, 2, 34) - G(2, 1, 34) \\ & + G(1, 23, 4) + G(2, 13, 4) + G(1, 24, 3) + G(2, 14, 3)] . \end{aligned} \quad (20)$$

This representation holds for the non-forward direction as well. After Fourier transformation to configuration space, the function  $G$  was shown to be conformally invariant by itself. The proof uses essentially the same method as the original one for  $V$  in [18], now applied to the simpler function  $G$ . Again one defines an amplitude  $\mathcal{A}_3$  in analogy to  $\mathcal{A}_4$  in (17), now with a three-gluon amplitude  $\phi_3$  instead of  $\phi_4$  and without the contraction in color space. The Fourier transform is then defined in analogy to (18) and (19). The most difficult step is to show the invariance under inversion,  $\rho \rightarrow 1/\rho$ . The explicit proof is rather technical and requires a careful treatment of the necessary regularization parameters (for details see [13, 14, 18]). A subtle point is the possible occurrence of infrared logarithms which can potentially break the conformal invariance of transition kernels in impact parameter space. Such logarithms can arise if the kernels do not vanish when their momentum arguments tend to zero. The function  $G$  does indeed not vanish when its second momentum argument vanishes, as we have seen in eq. (14), and one could thus expect dangerous infrared logarithms to arise. But it turns out that in the unitarity corrections  $G$  occurs only in combinations which have the property to vanish for vanishing momentum arguments. The function  $V\phi_2$  in the two-to-four gluon vertex for example is such a combination, see eq. (20). One can easily derive that

$$(V\phi_2)(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4)|_{\mathbf{k}_i=0} = 0 \quad (i \in \{1, \dots, 4\}) . \quad (21)$$

This implies that the potentially dangerous logarithms cancel in the vertices of the effective field theory of unitarity corrections and do not need to be considered any further. With this additional piece of information the conformal invariance of the vertex  $V_{2\rightarrow 4}$  can then be derived easily because it is a superposition of conformally invariant functions  $G$ .

The explicit expression for the function  $G$  in impact parameter space can be found in [13, 14] and will not be needed here. Unfortunately, it was so far not possible to find a representation for  $G$  which is of similar simplicity as (3) for the BFKL kernel, namely in terms of pseudodifferential operators. In particular, it appears that  $G$  and the vertex  $V_{2 \rightarrow 4}$  actually lack the property of holomorphic separability [12].

### 3 The two-to-six gluon vertex

We now turn to the two-to-six gluon vertex derived in [10]. Again we consider its action on a two-gluon state  $\phi_2$ . Let  $\mathbf{k}_i$  be the transverse momenta of the six produced gluons, and let  $a_i$  denote the corresponding color labels. Then the vertex has the form

$$(V_{2 \rightarrow 6}^{a_1 a_2 a_3 a_4 a_5 a_6} \phi_2)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6) = \sum d_{a_1 a_2 a_3} d_{a_4 a_5 a_6} (W \phi_2)(1, 2, 3; 4, 5, 6), \quad (22)$$

where  $d_{abc}$  denotes the symmetric structure constant of  $su(N_c)$ , and we again use the shorthand notation for the momentum arguments introduced below eq. (12). The sum extends over all (ten) partitions of the six gluons into two groups containing three gluons each,

$$\begin{aligned} & \sum d_{a_1 a_2 a_3} d_{a_4 a_5 a_6} (W \phi_2)(1, 2, 3; 4, 5, 6) = \\ & = d_{a_1 a_2 a_3} d_{a_4 a_5 a_6} (W \phi_2)(1, 2, 3; 4, 5, 6) \\ & \quad + d_{a_1 a_2 a_4} d_{a_3 a_5 a_6} (W \phi_2)(1, 2, 4; 3, 5, 6) + \dots \\ & \quad + d_{a_1 a_5 a_6} d_{a_2 a_3 a_4} (W \phi_2)(1, 5, 6; 2, 3, 4). \end{aligned} \quad (23)$$

The symmetry properties of the vertex are very similar to those of the two-to-four vertex  $V_{2 \rightarrow 4}$  discussed earlier. The function  $(W \phi_2)(1, 2, 3; 4, 5, 6)$  is completely symmetric in the first three gluon momenta as well as in the last three momenta, hence the notation. Further,  $W \phi_2$  is symmetric under the exchange of its first three with its last three arguments,

$$(W \phi_2)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6) = (W \phi_2)(\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \quad (24)$$

From (22) then follows the symmetry of the full vertex  $V_{2 \rightarrow 6}$  in the six gluons, i.e. the symmetry under the simultaneous exchange of momenta and color labels of the gluons. In [10] the function  $W \phi_2$  was calculated in terms of the integrals  $a, b, c, s$ , and  $t$ . The corresponding expression is rather long and we do not give it explicitly here.

The transformation to impact parameter space is performed as in the case of  $V_{2 \rightarrow 4}$ . An amplitude  $\mathcal{A}_6$  is defined similarly to  $\mathcal{A}_4$  in eq. (17). Now we have  $V_{2 \rightarrow 6}$  instead of  $V_{2 \rightarrow 4}$ , and  $\phi_4$  in that equation is replaced by a conformally invariant state  $\phi_6$  of six reggeized gluons solving the six-reggeon BKP equation. Obviously

the integration is extended to all six momenta  $\mathbf{k}_i$ . The Fourier transformation is then performed in analogy to eqs. (18) and (19) to get the vertex in impact parameter space. The most convenient way to prove its invariance under conformal transformations (4) is again to show that it can be written as a sum of  $G$ -functions already in momentum space, and thus also in impact parameter space. Such a representation for  $W\phi_2$  can in fact be found,

$$\begin{aligned}
(W\phi_2)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6) &= \frac{g^4}{8} \times \\
&[ G(123, -, 456) \\
&- G(12, 3, 456) - G(13, 2, 456) - G(23, 1, 456) \\
&- G(123, 4, 56) - G(123, 5, 46) - G(123, 6, 45) \\
&+ G(1, 23, 456) + G(2, 13, 456) + G(3, 12, 456) \\
&+ G(123, 45, 6) + G(123, 46, 5) + G(123, 56, 4) \\
&+ G(12, 34, 56) + G(13, 24, 56) + G(23, 14, 56) \\
&+ G(12, 35, 46) + G(13, 25, 46) + G(23, 15, 46) \\
&+ G(12, 36, 45) + G(13, 26, 45) + G(23, 16, 45) \\
&- G(1, 234, 56) - G(2, 134, 56) - G(3, 124, 56) \\
&- G(1, 235, 46) - G(2, 135, 46) - G(3, 125, 46) \\
&- G(1, 236, 45) - G(2, 136, 45) - G(3, 126, 45) \\
&- G(12, 345, 6) - G(12, 346, 5) - G(12, 356, 4) \\
&- G(13, 245, 6) - G(13, 246, 5) - G(13, 256, 4) \\
&- G(23, 145, 6) - G(23, 146, 5) - G(23, 156, 4) \\
&+ G(1, 2345, 6) + G(2, 1345, 6) + G(3, 1245, 6) \\
&+ G(1, 2346, 5) + G(2, 1346, 5) + G(3, 1246, 5) \\
&+ G(1, 2356, 4) + G(2, 1356, 4) + G(3, 1256, 4)]. \tag{25}
\end{aligned}$$

It is straightforward to prove this using the definition of  $G$ , see eq. (12). After some cancellations one obtains the original form of the vertex in terms of the integrals  $a, b, c, s$  and  $t$  as it was given in [10].

With the new representation for  $W\phi_2$  and thus for the two-to-six gluon vertex  $V_{2 \rightarrow 6}$  we have established the conformal invariance of the latter in impact parameter space. With this result we can now conclude that the whole set of amplitudes of the GLLA with up to six  $t$ -channel gluons is conformally invariant. This is possible because in [10] it was shown that these amplitudes consist of only the following building blocks: states of  $n$  reggeized gluons with  $n = 2, 4, 6$ , two-to-four transition vertices that can be expressed in terms of  $V$ , and a two-to-six transition given in terms of  $W$ . All of these elements, including  $W$ , are conformally invariant in impact parameter space.

A closer look at the expressions (20) and (25) reveals an interesting regularity in the construction of the functions  $V$  and  $W$  which define the two-to-four and



the two-to-six gluon vertex, respectively. In the remaining part of this section we want to study this regularity.

For simplicity of notation we again consider the action of  $V$  and  $W$  on the two-gluon state  $\phi_2$ . We recall that the function  $(V\phi_2)(\mathbf{k}_1, \mathbf{k}_2; \mathbf{k}_3, \mathbf{k}_4)$  is symmetric in its first two arguments as well as in its last two arguments. This matches the symmetry of the color structure it comes with in the two-to-four vertex  $V_{2\rightarrow 4}$ , see eq. (16). Similarly, the function  $(W\phi_2)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6)$  is completely symmetric in its first three as well as in its last three arguments. Again, this matches the symmetry of the color structure with which this permutation of arguments occurs in the full vertex, see eq. (22).

Let us now look in more detail at the arguments of the  $G$ -functions as they appear in the new representation for  $W\phi_2$  in eq. (25). The first term is  $G(123, -, 456)$  which equals  $(\mathcal{K}\phi_2)(123, 456)$  (up to a factor  $N_c$ ). All other terms in eq. (25) can now be constructed from this term by a simple procedure. For each possible subset of the six momenta  $\mathbf{k}_i$  we obtain exactly one term. These respective terms have the sum of the momenta in that subset as the second argument of the function  $G$ . At the same time, those momenta are taken out of the sums in the first and third argument of  $G$ . The sign of the resulting term alternates with the number of elements, i. e. the sign equals  $(-1)^l$  with  $l$  being the number of elements of the respective subset of momenta.

It appears useful to go through the above procedure in some examples. The simplest non-trivial subset of the six momenta is one containing only one element, say  $\mathbf{k}_1$ . Starting from  $G(123, -, 456)$  we put  $\mathbf{k}_1$  into the second argument of  $G$  while removing it from the first argument. With the minus sign ( $l = 1$ , see above) we have  $-G(23, 1, 456)$ . Similar terms are obtained for the other five momenta. There are 15 subsets of the six momenta that contain exactly two momenta. Let us consider the subset  $\{\mathbf{k}_1, \mathbf{k}_4\}$ . Removing  $\mathbf{k}_1$  from the first and  $\mathbf{k}_4$  from the third argument of  $G(123, -, 456)$  and taking their sum as the second argument we have  $G(23, 14, 56)$  which comes with a positive sign since now  $l = 2$ . For  $l \geq 3$  there are subsets which contains all of the momenta appearing in the first or third argument of  $G(123, -, 456)$ , for example  $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$ . In these cases the resulting  $G$ -function would have a vanishing first (or third) argument, and thus vanish itself due to (13).

It is easily checked that this procedure reproduces the correct sign and momentum structure of all terms in the new representation (25) of the function  $W\phi_2$ . Let us recall that the term that we started with,  $G(123, -, 456)$ , has exactly the same symmetry properties as the full function  $W\phi_2$ . This is consistent with the fact that the above procedure is completely symmetric in the six gluon momenta.

The function  $V\phi_2$  as given in eq. (20) can be obtained by the same procedure, now starting from  $G(12, -, 34)$ . We have thus found a general rule for constructing the known two-to-four and two-to-six gluon vertices in the effective field theory of unitarity corrections. The starting points are the terms  $G(12, -, 34)$  and  $G(123, -, 456)$ , respectively. Moreover, that rule even works for the special

case of the two-to-two transition vertex as which the BFKL kernel can be interpreted. In that case the term to start with is  $G(1, -, 2)$  and one can immediately apply eq. (14).

Remarkably, this procedure leads to vertex functions  $V\phi_2$  and  $W\phi_2$  which vanish if *any* outgoing momentum vanishes. For  $V\phi_2$  we have seen this already in eq. (21). Similarly, for  $W\phi_2$  we have

$$(W\phi_2)(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6)|_{\mathbf{k}_i=0} = 0 \quad (i \in \{1, \dots, 6\}). \quad (26)$$

This is in contrast to the function  $G$  from which they are constructed, see eq. (14). In [11] it was discussed that this condition appears to be characteristic for all possible transition vertices of the effective field theory of unitarity corrections.

It is now very suggestive to guess that a potential two-to-eight gluon vertex  $V_{2 \rightarrow 8}$  would be constructed in a similar way. Obviously, the corresponding color structure — the analogue of (22) — is unknown. The corresponding color decomposition will presumably lead to a function  $X$  with symmetry properties similar to  $V$  and  $W$ , namely  $X$  will be symmetric in its first four momentum arguments and in its last four momentum arguments. Its conjectured decomposition in terms of  $G$ -functions is obtained by the above rule starting from  $G(1234, -, 5678)$ . We expect that the effective field theory of unitarity corrections contains further vertices  $V_{2 \rightarrow 2m}$  ( $m \in \mathbb{N}$ ) to which the same procedure applies.

The computation of such vertices using the conventional method (see [10]) is extremely tedious. It would therefore be very interesting if our conjecture could be tested using other approaches to the problem of perturbative unitarity corrections.

We should add that the rule given here fixes the vertex functions only up to their overall normalization. The overall factors of the coupling constant  $g$  are adjusted in such a way that the power of  $g$  equals the number of outgoing gluons (having in mind that  $G$  already contains a factor  $g^2$ ). The numerical factors of  $1/2$  and  $1/8$  in eqs. (20) and (25), however, cannot be explained unambiguously at the moment. Another very important piece of information that can at the moment only be obtained by the full computation of the vertices is their color structure.

## 4 Summary

The perturbative unitarity corrections in the GLLA can be cast into the form of an effective field theory of interacting  $n$ -reggeized gluon states in the  $t$ -channel and transition vertices connecting states with different numbers of reggeized gluons. We have studied one of the building blocks of this effective field theory, namely the two-to-six reggeized gluon vertex. A new representation for this vertex was found which proves its invariance under global conformal (Möbius) transformations in two-dimensional impact parameter space. With the help of this representation we have also been able to find a regularity in the expressions for the two-to-four and the two-to-six transition vertices. This regularity might be a helpful observation

for the further investigation of the elements of the effective field theory of unitarity corrections.

By proving the conformal invariance of the two-to-six transition vertex we have established that the full set of amplitudes in the GLLA with up to six reggeized gluons in the  $t$ -channel is conformally invariant. This lends further support to the conjecture that the whole set of unitarity corrections in the GLLA can be formulated as a conformal field theory in two-dimensional impact parameter space with rapidity as an additional real parameter. The still open and very challenging problem is of course to identify this conformal field theory.

Eventually one would like to consider the unitarity corrections in next-to-leading logarithmic approximation (NLLA). So far only the NLL corrections to the BFKL Pomeron are known [21, 22]. One expects that the unitarity corrections will exhibit the structure of an effective field theory also in the generalized NLLA. It would be desirable to compute the NLL corrections to all elements of the effective field theory. This is an extremely difficult problem, and so far the only step in this direction has been to make an educated guess for the two-to-four gluon vertex in NLLA [15]. It is known that the NLL corrections to the BFKL Pomeron break the conformal invariance in impact parameter space. But it turns out that this breaking is soft in the sense that it occurs only due to the running of the gauge coupling. One hopes that this will be true also for the other elements of the unitarity corrections in the generalized NLLA. In this case the conformal invariance would remain an extremely powerful tool for the investigation of unitarity corrections in high energy QCD.

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